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The Dynamics of Population Growth, Differential Fertility, and Inequality: Comment

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In a recent issue of this *Review*, David Lam (1986) examined the relationship between population growth and the distribution of income. The model that was applied, allowed for differential fertility across income classes and intergenerational mobility between them. Lam claimed that if $M_{ii} > M_{ij}$ for all $j \neq i$, then an increase in the fertility of income class i will increase the percentage of the population in income class i in the steady state (p. 1110). M_{ij} denotes the probability that a child of class j becomes a member of class i. By means of a counterexample, C. Y. Cyrus Chu (1987) has shown that Lam's proposition is not true; the steady-state proportion of class i may fall.

Intuitively, one could argue that, when the decrease in the steady-state proportion of class i is large enough, also the percent children in class i will fall, despite the increased fertility in this class. In the present note we show that precisely this is impossible. So, in spite of a potential reduction in the relative size of the class with increased fertility, the proportion of the steady-state population born to that class will always increase and, moreover, will increase relatively the most. As a direct consequence, the possible decrease in the steady-state proportion of income class i is limited. The use of matrix algebra instead of differential calculus, enables us to derive upper and lower bounds for the relative change in the proportion of the steady-state population of any income class. Surprisingly, the results are obtained without any restrictions on the mobility matrix $\mathbf{M}$, therefore they simply are properties of the model.

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I. The Model

Let the n-column vector $\mathbf{P}_t$ denote the size of the income classes in period t. The diagonal matrix $\mathbf{F}$ gives the income-specific net reproduction rates (F_i) and $\mathbf{M} = [M_{ij}]$ is the intergenerational mobility matrix. Thinking of each period as a generation, the distribution of income in period t is characterized by

$$\mathbf{P}_t = \mathbf{MFP}_{t-1}. \tag{1}$$

Let the total population size at time t be given by $N_t \equiv \mathbf{e}'\mathbf{P}_t$, with $\mathbf{e}'$ the n-row summation vector, that is, $\mathbf{e}' \equiv [1,\dots,1]$. Denote the population growth rate in period t with $g_t \equiv N_t/N_{t-1}$. Let the elements of the n-column vector π_t denote the proportions of the population in each income class; then $\boldsymbol{\pi}_t = \mathbf{P}_t/N_t$. Dividing both sides of (1) by N_{t-1} yields

$$g_t\boldsymbol{\pi}_t = \mathbf{P}_t/N_{t-1} = \mathbf{MFP}_{t-1}/N_{t-1} = \mathbf{MF}\boldsymbol{\pi}_{t-1}. \tag{2}$$

The proportions of the tth period population born to parents from the different income classes are given by the elements of the n-column vector $\boldsymbol{\gamma}_t \equiv \mathbf{FP}_{t-1}/N_t$. Thus $N_t\boldsymbol{\gamma}_t = \mathbf{FP}_{t-1}$, dividing both sides by N_{t-1} yields $g_t\boldsymbol{\gamma}_t = \mathbf{F}\boldsymbol{\pi}_{t-1}$ or

$$\boldsymbol{\gamma}_t = \mathbf{F}\boldsymbol{\pi}_{t-1}/g_t. \tag{3}$$

This vector shows the distribution of the offspring before the transition between the income groups takes place (for $\boldsymbol{\pi}_t = \mathbf{M}\boldsymbol{\gamma}_t$). Note that the intergenerational mobility matrix $\mathbf{M}$ is a transition matrix so that $\mathbf{e}'\mathbf{M} = \mathbf{e}'$. The definitions and the equations above imply the following identities: $\mathbf{e}'\boldsymbol{\pi}_t = 1$, $\mathbf{e}'\mathbf{F}\pi_{t-1} = g_t$ and $\mathbf{e}'\boldsymbol{\gamma}_t = 1$. In the steady state $\boldsymbol{\pi}_t = \boldsymbol{\pi}$ and $g_t = g$, implying $\boldsymbol{\gamma}_t = \boldsymbol{\gamma}$. Dropping the time subscripts, (2) and (3) are rewritten as

$$\mathbf{MF}\boldsymbol{\pi} = g\boldsymbol{\pi}, \tag{4}$$

$$\boldsymbol{\gamma} = \mathbf{F}\boldsymbol{\pi}/g. \tag{5}$$

Consequently, the identities in the steady state become $\mathbf{e}'\boldsymbol{\pi} = 1$, $\mathbf{e}'\mathbf{F}\boldsymbol{\pi} = g$ and $\mathbf{e}'\boldsymbol{\gamma} = 1$. As usual it is assumed that the matrix **MF** is irreducible and primitive, in order to ensure convergence. Then it follows from the Perron-Frobenius theorem that $g > 0$ is the dominant eigenvalue of **MF** and $\boldsymbol{\pi} \gg 0$[1] the corresponding eigenvector, which is unique after normalization ($\mathbf{e}'\boldsymbol{\pi} = 1$).

II. The Results

Suppose that the fertility of income class i is increased and denote its new reproduction rate with $\bar{F}_i$ ($> F_i$). All other reproduction rates remain unchanged, that is $\bar{F}_j = F_j$ for all $j \neq i$. (4) and (5) are, for the new steady state, written as

$$\mathbf{M\bar{F}}\bar{\boldsymbol{\pi}} = \bar{g}\bar{\boldsymbol{\pi}}, \tag{6}$$

$$\bar{\boldsymbol{\gamma}} = \bar{\mathbf{F}}\bar{\boldsymbol{\pi}}/\bar{g}. \tag{7}$$

The following ordering of matrices holds: $\mathbf{MF} < \mathbf{M\bar{F}} < (\bar{F}_i/F_i)\mathbf{MF}$. From the Perron-Frobenius theorem it is known that the same ordering applies to the respective dominant eigenvalues.[2] Thus $g < \bar{g} < (\bar{F}_i/F_i)g$, or $1 < \bar{g}/g < \bar{F}_i/F_i$.

PROPOSITION 1: *$\bar{\pi}_j/\pi_j \leq g\bar{F}_i\bar{\pi}_i/\bar{g}F_i\pi_i$ for all j.*

PROOF:

The assertion obviously holds for $j = i$. Now, suppose to the contrary that there exists an index $k \neq i$ such that

$$\frac{\bar{\pi}_k}{\pi_k} = \max_{j \neq i} \frac{\bar{\pi}_j}{\pi_j} > \frac{g\bar{F}_i\bar{\pi}_i}{\bar{g}F_i\pi_i}.$$

Then:

$$\begin{aligned} g\bar{\pi}_k &= (g/\bar{g})\bar{g}\bar{\pi}_k = (g/\bar{g})\sum_{j=1}^{n} M_{kj}\bar{F}_j\bar{\pi}_j \\ &= \sum_{j\neq i}(g/\bar{g})M_{kj}F_j\bar{\pi}_j + (g/\bar{g})M_{ki}\bar{F}_i\bar{\pi}_i \\ &= \sum_{j\neq i}(g/\bar{g})(\bar{\pi}_j/\pi_j)M_{kj}F_j\pi_j \\ &\quad + (g/\bar{g})(\bar{F}_i\bar{\pi}_i/F_i\pi_i)M_{ki}F_i\pi_i \\ &< (\bar{\pi}_k/\pi_k)\sum_{j=1}^{n} M_{kj}F_j\pi_j = g\bar{\pi}_k, \end{aligned}$$

which is not possible. Strict inequality holds because: (i) $g < \bar{g}$; (ii) $\bar{\pi}_j/\pi_j \leq \bar{\pi}_k/\pi_k$ for all j; (iii) $(g/\bar{g})(\bar{F}_i\bar{\pi}_i/F_i\pi_i) < \bar{\pi}_k/\pi_k$ by assumption, and (iv) $M_{kj} > 0$ for some j, because of the irreducibility of **M**. We have thus obtained a contradiction. □

This proposition enables us to derive upper and lower bounds for the relative changes in the steady-state proportions of the income classes. The right-hand side of the expression in the proposition equals $\bar{\gamma}_i/\gamma_i$ and from $\bar{g}/g < \bar{F}_i/F_i$ follows that it is larger than $\bar{\pi}_i/\pi_i$. Applying the proposition to (5) and (7), using $\bar{F}_j = F_j$ for $j \neq i$, yields that the following inequality holds for all $j \neq i$.

$$\bar{\gamma}_j/\gamma_j = (g/\bar{g})(\bar{\pi}_j/\pi_j) \leq [(g/\bar{g})(\bar{\gamma}_i/\gamma_i) < (\bar{\gamma}_i/\gamma_i)]. \tag{8}$$

First, equation (8) yields $\bar{\pi}_j/\pi_j = (\bar{g}/g)(\bar{\gamma}_j/\gamma_j) > \bar{\gamma}_j/\gamma_j$. Second, $\mathbf{e}'\bar{\boldsymbol{\gamma}} = \mathbf{e}'\boldsymbol{\gamma} = 1$ and $\bar{\gamma}_j/\gamma_j < \bar{\gamma}_i/\gamma_i$ for all $j \neq i$ from (8), imply $\bar{\gamma}_i > \gamma_i$. Consequently, the right-hand side of the expression in the proposition is larger than one or, equivalently, $\bar{\pi}_i/\pi_i > (\bar{g}/g)(F_i/\bar{F}_i)$. The bounds are summarized by the following inequalities.

[1] For vectors and matrices we adopt the following notations. $\mathbf{x} \geq \mathbf{y}$ means $x_i \geq y_i$ for all i, $\mathbf{x} > \mathbf{y}$ means $\mathbf{x} \geq \mathbf{y}$ but $\mathbf{x} \neq \mathbf{y}$, and $\mathbf{x} \gg \mathbf{y}$ means $x_i > y_i$ for all i.

[2] See for instance Samuel Karlin and Howard Taylor (1975, Appendix 2) or Akira Takayama (1985, Chapter 4).

(9) $(\bar{g}/g)(F_i/\bar{F}_i) < \bar{\pi}_i/\pi_i$

$< \bar{\gamma}_i/\gamma_i$ and $\bar{\gamma}_i > \gamma_i$.

(10) $\bar{\gamma}_j/\gamma_j < \bar{\pi}_j/\pi_j \leq \bar{\gamma}_i/\gamma_i$ for all $j \neq i$.

III. Conclusions and an Example

The central result appears to be $\bar{\gamma}_i > \gamma_i$. If the fertility of income class i is increased, then the proportion of the population born to parents from class i will increase in the steady state. Lam's proposition that "an increase in the fertility of the poor will unambiguously increase the percent poor in the steady state" holds for the percent children of the poor. At first sight, this result may seem obvious. From Chu's example however, it is known that the steady-state proportion of income class i may fall. When the decrease in the steady-state proportion of class i is sufficiently large, it follows from $\gamma_i = F_i\pi_i/g$ that the percent children in class i must also fall. Precisely this, has been shown to be impossible. An immediate consequence of this result is that the possible decrease in the steady-state proportion of class i is bounded from below.[3] The lower bound in (9) therefore, tells us the degree to which Lam's original proposition might go wrong.

Equation (10) asserts that the steady-state proportion of the children increases for income class i more than for any other income class. Furthermore it is known that for some income class (other than i) it must fall. Given these two results, that is, $\bar{\gamma}_i > \gamma_i$ and $\bar{\gamma}_j/\gamma_j < \bar{\gamma}_i/\gamma_i$ for all $j \neq i$, the upper bounds in (9) and (10) are easily explained. The proportion of the population in income class k, is composed of all children that have become a member of class k, that is, $\pi_k = \Sigma_{j=1}^{n} M_{kj}\gamma_j$. The influx from any class other than i, increases less than $\bar{\gamma}_i/\gamma_i$. Therefore, $\bar{\pi}_k/\pi_k \leq \bar{\gamma}_i/\gamma_i$ for $k \neq i$[4] and $\bar{\pi}_i/\pi_i < \bar{\gamma}_i/\gamma_i$.[5]

Table 1—Steady-State Proportions and Ratios for Chu's Example

	$j=1$	$j=2$	$j=3$
π_j	0.1871	0.4516	0.3613
$\bar{\pi}_j$	0.1870	0.4507	0.3623
$\bar{\pi}_j/\pi_j$	0.9996	0.9980	1.0027
$\bar{\gamma}_j$	0.1885	0.4499	0.3616
$\bar{\gamma}_j/\gamma_j$	1.0077	0.9961	1.0008

The lower bound in (10) provides another assertion that is difficult to explain intuitively. If the percent children in any income class $j \neq i$ increases, then the proportion of the population after intervention of mobility increases also. We know that the inflows from classes i and j rise, while the influx from at least one class falls. Apparently, the decreased inflows can never outweigh the increased inflows, in this case ($\bar{\gamma}_j > \gamma_j$).

Once again, it should be noted that the results hold without any restrictions on the mobility matrix, as such they simply are properties of the model. As an illustration of the results, we finally consider Chu's counterexample,

$$\mathbf{M} = \begin{bmatrix} 0.3 & 0.29 & 0 \\ 0 & 0.6 & 0.5 \\ 0.7 & 0.11 & 0.5 \end{bmatrix}.$$

$(F_1, F_2, F_3) = (1,1,1)$ and clearly $g = 1$. The reproduction rate of income class 1 is raised by 1 percent ($\bar{F}_1 = 1.01$), which yields for the population growth rate $\bar{g} = 1.00187$, while the lower bound in (9) turns out to be 0.99195. The calculated values for the various proportions and ratios are presented in Table 1, rounded off to four decimals. Note that $i = 1$ and $\gamma = \pi$. The bounds in (9) and (10) are easily checked to hold.

[3] Note that the lower bound in (9) requires the computation of the dominant eigenvalues g and $\bar{g}$. From $\bar{g} > g$, it follows that $F_i/\bar{F}_i$ is also a lower bound for $\bar{\pi}_i/\pi_i$. Obviously, this latter bound is weaker, on the other hand however, it may be calculated even if case **M** is unknown.

[4] Equality holds only if all members of class k are born in class i, that is: $M_{kj} = 0$ for all $j \neq i$.

[5] Equality would necessarily imply $M_{ij} = 0$ for all $j \neq i$, which contradicts the irreducibility of **M**.

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